Young Researchers Workshop on Geometric Analysis, Singular PDEs and Numerics

RWTH Office West, 3rd floor, seminar room 377, Im Süsterfeld 2

Monday, June 1st

08:50-09:00 Welcome

09:00-10:00 On well-posedness of s-Schroedinger maps
Armin Schikorra

10:00-10:30 Coffee Break

10:30-11:30 A posteriori error estimates for a modified Morley FEM
Mira Schedensack

11:30-12:00 Homogenization with Slender Bodies
Pascal Hadré

12:00-14:00 Lunch Break

14:00-14:30 Thresholding Scheme for the Half-Harmonic Map Heat Flow
Kilian Koch

14:30-15:00 On Hypoellipticity of Hörmander Operators
Jona Stahlschmidt

15:00-15:30 A functionalized-Cahn-Hilliard type model for two-phase flow with soluble surfactants
Clemens Ullrich

15:30 Open Discussions

18:00 Dinner

Tuesday, June 2nd

09:00-10:00 Analyticity of Solutions to Fractional Partial Differential Equations
Simon Blatt

10:00-10:30 Coffee Break

10:30-11:30 Monotone finite element approximation of second-order mean field game systems
Yohance Osborne

11:30-12:00 Circles minimize most tangent-point energies
Alexander Dohmen

12:00-14:00 Lunch Break

14:00-14:30 Homogenization of the Navier-Stokes equations in the inviscid limit in a randomly perforated domain
Eleni Hübner-Rosenau

14:30-15:00 Numerical Methods for a half-harmonic map heat flow problem
Nam Anh Nguyen

15:00 Open Discussions

Wednesday, June 3rd

09:00-10:00 Generalized convex hull for the coplanar n-well problem and its relation to pattern formation in thin-film shape memory alloys
Lauro Morales Montesinos

10:00-10:30 Coffee Break

10:30-11:30 On a complete Riemannian metric on the space of closed embedded curves
Philipp Reiter

11:30-12:00 Barotropic-Baroclinic Splitting for Multilayer Shallow Water Models
Sophie Hörnschemeyer

12:00-14:00 Lunch Break

14:00-14:30 Vortex motion for the mixed Ginzburg–Landau flow under stochastic perturbations
Valentin Linse

14:30-15:00 Symmetry and the number of critical knots
Nicolas Freches

15:00 Open Discussions

SFB 1481 Sparsity and Singular Structures
RWTH AACHEN UNIVERSITY
Energy, Entropy, and Dissipative Dynamics
DFG Deutsche Forschungsgemeinschaft

Abstracts

On well-posedness of s-Schroedinger maps

Armin Schikorra

University of Pittsburgh

Mon, 01 June, 9:00am

I am going to present recent progress on well-posedness of a nonlinear Schroedinger system with loss of derivatives that is a model equation for the s-Schroedinger map system $\partial_t u = u \wedge (-\Delta)^s u$ where u maps into the two-sphere $\mathbb{S}^2$ – For $s = 1/2$ is the halfwave map equation, for $s = 1$ it is the Schroedinger map equation. Joint work with Ahmed Dughayshim and Silvino Reyes-Farina.

A posteriori error estimates for a modified Morley FEM

Mira Schedensack

Leipzig University

Mon, 01 June, 10:30am

This talk considers a singularly perturbed biharmonic equation. While the standard nonconforming Morley FEM works well for the biharmonic problem, it diverges for the Poisson problem. It therefore does not converge robustly in the perturbation parameter for the singularly perturbed biharmonic equation.

The modified Morley FEM proposed in [J. Comput. Math, 24(2), 2006] introduces the nodal interpolation operator in the Laplace term to overcome this. This talk derives residual-based a posteriori error estimators for this method. The error estimators are proven to be reliable and efficient. Moreover, an adaptive algorithm driven by these error estimators is investigated in numerical experiments.

Homogenization with Slender Bodies

Pascal Hadré

University Regensburg

Mon, 01 June, 11:30am

In this talk, we aim to derive macroscopic equations for suspensions of slender bodies. A slender body is a rigid particle of length a_ϵ and width b_ϵ, where $b_\epsilon \ll a_\epsilon$ and $a_\epsilon \to 0$ as $\epsilon \to 0$. More precisely, we focus on the case in which these particles are fixed and distributed on a periodic grid of size $\epsilon > 0$. We determine the critical scaling for a_ϵ and b_ϵ such that the solutions (u_ϵ, p_ϵ) of the stationary Stokes equation in a domain perforated by the particles converge, as $\epsilon \to 0$, to a solution (u, p) of the stationary Stokes equation with an additional Brinkman term Mu, where $M \in \mathbb{R}^{3\times 3}$. We illustrate the steps required to rigorously set up this problem and present the main ideas of the proof, which follows the strategy introduced by Cioranescu and Murat. Moreover, we determine the structure of the matrix M and discuss possible generalizations and directions for future work.

Thresholding Scheme for the Half-Harmonic Map Heat Flow

Kilian Koch

RWTH Aachen

Mon, 01 June, 2:00pm

We study the half-harmonic map heat flow, the gradient flow of the half-Dirichlet energy for sphere-valued maps on the torus. Motivated by the thresholding algorithm of Laux and Yip for mean curvature flow, we replace the classical Gaussian kernel by the Poisson kernel, the semigroup kernel associated to the half-Laplacian, and alternate convolution with pointwise projection onto the sphere. We prove that the resulting iterates converge, along a subsequence, to a weak solution of the flow attaining the initial data in $H^{1/2}$ and satisfying an energy inequality. We also consider a truncated variant of the scheme, where the Poisson kernel is projected onto the first N Fourier modes. Here convergence holds under the condition that $hN - d(d + 1)\log(1/h) \to \infty$, meaning N must grow slightly faster than $\log(1/h)/h$ as $h \to 0$. This truncated version is intended as a first step toward a fully implementable numerical method. As a complement, we establish weak-strong uniqueness. A weak solution satisfying the energy inequality must coincide with any strong solution sharing the same initial data, showing in particular that the scheme converges to the strong solution whenever one exists.

On Hypoellipticity of Hörmander Operators

Jona Stahlschmidt

Max Planck Institute for Mathematics in the Sciences

Mon, 01 June, 2:30pm

A differential operator is called *hypoelliptic* if the smoothness of the right-hand side of the associated differential equation immediately implies the smoothness of its solutions. While this qualitative definition generalizes elliptic differential operators, it provides little direct insight into the actual structure of the operator itself.

In this talk, I will present a classical result that identifies a broad class of second-order linear differential operators - the so-called *Hörmander operators* - as hypoelliptic. I will also discuss some more recent developments related to this theory.

A functionalized-Cahn-Hilliard type model for two-phase flow with soluble surfactants

Clemens Ullrich

Friedirch-Alexander-University

Mon, 01 June, 3:00pm

The surface tension of a fluid interface can be reduced due to the presence of a surfactant (= 'surface active agent'). Surfactants are used in various industrial applications, e.g. as emulsifiers, detergents, or wetting agents. In this talk, I will investigate a phase-field model for two-phase flow of incompressible fluids in the presence of a soluble surfactant.

The free energy governing the evolution incorporates a Cahn-Hilliard energy weighted by a (not necessarily nonnegative) prefactor that depends on the surfactant density, a bending energy for the interface, and a bulk mixture energy. As a result, an increase in interfacial area may be energetically favorable. This free energy functional resembles a functionalized Cahn-Hilliard energy, thus giving rise to a sixth-order Cahn-Hilliard-Willmore type equation for the phase-field parameter. Mathematical challenges are posed by the low regularity of the surfactant density.

We prove the existence of weak solutions on arbitrary time intervals, assuming the two fluids to be shear-thickening generalized Newtonian of power-law type. The argument is based on a semi-implicit discretization and a solenoidal Lipschitz truncation technique to deal with the nonlinearity in the viscous stress tensor.

Analyticity of Solutions to Fractional Partial Differential Equations

Simon Blatt

Paris-Lodron-University Salzburg

Tue, 02 June, 9:00am

We will explore a classic topic in the realm of partial differential equations (PDEs) within a contemporary context: the analyticity of solutions to elliptic equations. While initial results for classical elliptic PDEs were established by Bernstein in 1904, the landscape for fractional and non-local equations remains less fully charted, with only partial results or findings pertaining to very specific cases, such as the Hartree-Fock and Boltzmann equations, available to date.

Monotone finite element approximation of second-order mean field game systems

Yohance Osborne

Durham University

Tue, 02 June, 10:30am

Mean Field Games (MFG) are models for Nash equilibria in games of optimal control involving a continuum of players. MFG are typically characterized by a system of PDEs which describes the nonlinear relationship between the optimal actions of individual players and the distribution of players in the game. The MFG system is given by a Hamilton—Jacobi—Bellman (HJB) equation, for the generic player's control optimization, that is coupled nonlinearly with a Kolmogorov—Fokker—Planck (KFP) equation that determines the non-negative player density.

MFG PDE systems have been studied extensively with regards to both their continuous analysis and approximation by numerical methods. However, attention has primarily been given to models where the players have access to unique optimal controls and the convex Hamiltonian of the system is differentiable. To model MFG where players have access to possibly non-unique optimal controls and the Hamiltonian is nondifferentiable, we recently proposed that the MFG PDE system be relaxed to a Partial Differential Inclusion (PDI) system based on measurable selections of the partial subdifferential of the Hamiltonian.

In this talk, we introduce monotone finite element methods for second-order MFG PDI systems with Lipschitz, nondifferentiable Hamiltonians. After motivating the MFG PDI system, we will present results concerning the existence and uniqueness of weak solutions for MFG PDI. We will then formulate a class of monotone finite element discretizations of MFG PDI, and prove the strong-norm convergence of the approximations together with rates.

Circles minimize most tangent-point energies

Alexander Dohmen

RWTH Aachen

Tue, 02 June, 11:30am

Joint work with Elias Döhrer (Chemnitz University of Technology)

Knot energies are functionals defined on the space of closed embedded space curves, that blow up whenever an embedding degenerates. The natural candidate for a global minimizer, when it exists, is the round circle. We show that for a broad range of parameters the circle uniquely minimizes the generalized tangent-point energies, a two-parameter family of knot energies, on the entire domain. Our proof relies on sharp lower bounds for the path length of certain Gauss maps associated with the curves, generalizing a previous argument by Simon Blatt for the classical tangent-point energies.

Homogenization of the Navier-Stokes equations in the inviscid limit in a randomly perforated domain

Eleni Hübner-Rosenau

University Regensburg

Tue, 02 June, 2:00pm

We study the behaviour of solutions to the Navier-Stokes equations with vanishing viscosity and a non-slip condition in a perforated domain. We consider the whole three-dimensional space where we remove N small holes that are randomly i.i.d. distributed. The behaviour depends on the particle size and on the viscosity of the fluid. We prove quantitative convergence results, provided that the local Reynolds number is small, in the subcritical and critical regime. In the first case, we obtain convergence to the Euler equations, whereas in the second case we obtain the Euler-Brinkman equations.

Numerical methods for a half-harmonic map heat flow problem

Nam Anh Nguyen

RWTH Aachen

Tue, 02 June, 2:30pm

Joint work with Kilian Koch, Christof Melcher and Arnold Reusken.

In recent years, the half-harmonic map heat flow mapping into the unit sphere has received significant interest in the analysis community on geometric PDEs. However, the numerical treatment of this problem is almost absent from the literature. The main difficulties are the non-local nature of the half Laplacian and the unit-length constraint of the solution.

In this talk, we present two methods that can handle these issues. The first is the thresholding scheme motivated by the work of Laux and Yip: First solve the linear fractional diffusion problem and then a subsequent projection step is applied. The second method relies on the orthogonality condition between the time derivative and its solution to impose the unit-length constraint which has been analyzed for the classical harmonic map heat flow and Landau-Lifshitz-Gilbert equation. This results in the construction of a discrete tangent space by a suitable spectral basis to approximate the time derivative and treat the half Laplacian efficiently. We investigate and compare these two methods in numerical experiments.

Generalized convex hull for the coplanar n-well problem and its relation to pattern formation in thin-film shape memory alloys

Lauro Morales Montesinos

National Autonomous University of Mexico

Wed, 03 June, 9:00am

In shape-memory alloys, it is common to analyze pattern formation induced by the existence of a finite number of zero-energy material phases $U \subset \mathbb{R}^{n\times n}$. These phases are characterized as minima of a non-convex bulk energy func- tional, $I(U)$, which exhibits scale-invariance. This invariance leads to the exis- tence of minimizing sequences which weakly converge to some "average" phases that are not minima of $I(U)$ and generate the observed microstructure in the sample. The set of all constant weak limits is known as the quasiconvex hull, QU. Determining QU is a challenging task, and only a few examples of sets U have explicit sets QU. In this talk, we investigate the set QU for a finite set $U \subset \mathbb{R}^{2\times 2}_{\text{sym}}$. We will identify an explicit set BU, contained in the convex hull CU of the wells, such that $QU \subset BU \subset CU$. Additionally, we will explore the conditions on the well distribution to attain the equality between QU and BU.

On a complete Riemannian metric on the space of closed embedded curves

Philipp Reiter

TU Chemnitz

Wed, 03 June, 10:30am

joint work with Elias Döhrer and Henrik Schumacher (Chemnitz University of Technology / RWTH Aachen University)

In pursuit of finding shortest paths in the manifold of closed embedded space curves we introduce a Riemannian metric which is inspired by the tangent-point potential, a self-repulsive functional. The latter blows up if an embedding degenerates which corresponds to infinite barriers between different isotopy classes.

For finite-dimensional Riemannian manifolds the Hopf—Rinow theorem guarantees that the Heine—Borel property (bounded sets are relatively compact), geodesic completeness (long-time existence of geodesic shooting), and metric completeness of the geodesic distance are equivalent. Moreover, it states that existence of length-minimizing geodesics follows from each of these statements. Albeit the Hopf—Rinow theorem does not hold true in this generality for infinite-dimensional Riemannian manifolds, we can prove all its four assertions for a suitably chosen Riemannian metric on the space of closed embedded curves.

Barotropic-Baroclinic Splitting for Multilayer Shallow Water Models

Sophie Hörnschemeyer

RWTH Aachen

Wed, 03 June, 11:30am

Multilayer ocean models are popular approximations to the 3D Euler and Navier-Stokes equations. Computational cost obviously increases with the number of layers, which is often chosen to be around 50 in ocean simulations. The barotropic-baroclinic splitting is an important strategy used in numerical ocean models to reduce this computational cost.

In the present contribution, we focus on the numerical analysis of the barotropic-baroclinic splitting in the context of finite volume schemes. We reformulate the splitting strategy within the nonlinear multilayer framework using terrain-following coordinates, and present it as an exact operator splitting. The barotropic step captures the evolution of free surface and depth averaged velocity with a well-balanced one-layer shallow water model. The baroclinic step incorporates vertical exchanges between layers and adjusts velocities around their mean vertical value.

Our version of the splitting is numerically robust, i.e. no filters or corrections are needed. The numerical solution inherently observes a discrete maximum principle for the tracer and hence guarantees non-negative tracer concentrations. In the language of applied mathematics, we prove a discrete entropy inequality. In the language of geophysics, this guarantees dissipation of kinetic and potential, and therefore of total energy. This is the key stability property for the class of finite volume schemes under consideration. Last, but not least, the gain in terms of computational cost is large, especially in low Froude simulations.

Currently, this work addresses the constant density case; however, ongoing work extends the barotropic-baroclinic splitting to variable density scenarios and models situations such as coastal upwelling.

Vortex motion for the mixed Ginzburg–Landau flow under stochastic perturbations

Valentin Linse

RWTH Aachen

Wed, 03 June, 2:00pm

The mixed Ginzburg–Landau flow $(\alpha_\varepsilon + i)\,\partial_t u_\varepsilon = -(\Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon(1 - |u_\varepsilon|^2)$ interpolates between gradient and Hamiltonian dynamics; in the singular limit $\varepsilon \to 0$ its vortices obey the motion law of Kurzke–Marzuola–Melcher–Spirn. In this talk we discuss how this picture changes under a rough, gauge-type (modulus-preserving) perturbation $i\, u_\varepsilon \sum_k g_k \circ \mathrm{d}B_k$, driven by a Q-Wiener process or a geometric rough path.

Symmetry and the number of critical knots

Nicholas Freches

RWTH Aachen

Wed, 03 June, 2:30pm

Building on a multiplicity result by Sakuma concerning symmetries of certain knot classes in S^3, we show how these symmetries can be transferred to symmetries of knots in $\mathbb{R}^3$. Using this correspondence together with a symmetry-preserving Sobolev gradient flow, we derive a lower bound for the number of critical points of certain knot energies. Along the lines, we also obtain a lower bound for the equivariant Lusternik–Schnirelmann category of the corresponding knot classes in $\mathbb{R}^3$. In particular, we show that there exist knot classes with arbitrarily many critical points.
